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INCOMPLETE BLOCK DESIGNS FOR COMPARING TREATMENTS WITH A CONTROL, III. OPTIMAL DESIGNS FOR p = 4, k = 3 AND p = 5, k = 3

by

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ABSTRACT

This paper continues the study of balanced treatment incomplete block (BTIB) designs. ~~initiated in [1] and [2]~~. The definitions of inadmissibility and strong inadmissibility ~~introduced in [2]~~ are generalized. The search for optimal designs is restricted to the smaller class of admissible designs obtained using these new definitions. Lists of generator designs, the (conjectured) complete class of generator designs, catalogs of admissible designs, and tables of optimal designs are given for $p = 4$, $k = 3$ and $p = 5$, $k = 3$.

Key words and phrases: Multiple comparisons with a control, balanced treatment incomplete block (BTIB) designs, admissible designs, S-inadmissible designs, C-inadmissible designs, complete class of generator designs, optimal designs.

1. INTRODUCTION AND SUMMARY

This paper continues the study of balanced treatment incomplete block (BTIB) designs which was initiated in [1] and [2]. This class of designs is appropriate for comparing simultaneously $p \geq 2$ test treatments with a control treatment (the so-called multiple comparisons with a control (MCC) problem) when the observations are taken in incomplete blocks of common size $k < p+1$. The reader is referred to [1] and [2] for background, motivation, and notation.

In [2] we studied and gave optimal designs for the cases $p = 2$, $k = 2(1)6$ and $p = 3, k = 3$; in the present paper we do the same for the cases $p = 4, k = 3$ and $p = 5, k = 3$. These later cases are markedly different from the former ones: For each of the six cases studied in the earlier paper it is known that there are only <u>two</u> (non-equivalent) admissible generator designs while for the two cases studied in the present paper there are <u>many</u> (nonequivalent) admissible generator designs. The full set of generator designs is not known for either of these two cases, and the optimal designs that we give are <u>optimal relative to the generator designs which are known to us</u>; however, we <u>conjecture</u> that we have enumerated <u>all</u> of the admissible generator designs for each of these two cases, and that if additional ones do exist the incremental gain that would be achieved by using the full set in place of our set would be very small. In the present paper it was found necessary to define admissibility in a more restricted sense than in the earlier paper, and some designs which would have been admissible under our old definition are no longer admissible under our new definitions. The new

definitions of inadmissibility are given in Section 2; these new definitions permit us to narrow down the number of candidates which are eligible to be considered as optimal designs.

As in [2] we are interested in making exact joint one-sided confidence interval estimates of the form

$$\{\alpha_0 - \alpha_i \geq \hat{\alpha}_0 - \hat{\alpha}_i - d \quad (1 \leq i \leq p)\} \tag{1.1}$$

for the treatment differences $\alpha_0 - \alpha_i$ based on their BLUE's $\hat{\alpha}_0 - \hat{\alpha}_i$ $(1 \leq i \leq p)$ for given values of (p,k,b) when σ^2 is known, and $d > 0$ is a specified "yardstick." We limit consideration to designs (which we refer to as BTIB designs) for which

$$\begin{aligned} &\mathrm{Var}\{\hat{\alpha}_0 - \hat{\alpha}_i\} = \tau^2\sigma^2 \quad (1 \leq i \leq p) \\ &\mathrm{Corr}\{\hat{\alpha}_0 - \hat{\alpha}_{i_1}, \hat{\alpha}_0 - \hat{\alpha}_{i_2}\} = \rho \quad (i_1 \neq i_2;\ 1 \leq i_1, i_2 \leq p); \end{aligned} \tag{1.2}$$

such designs can be characterized in terms of two parameters (λ_0, λ_1) defined in Theorem 3.1 of [1]. An expression for $\hat{\alpha}_0 - \hat{\alpha}_i$ is given by (2.1) of [2]. The confidence coefficient (P) associated with (1.1) when BTIB designs are used can be written as

$$P = \int_{-\infty}^{\infty} \left[\Phi\left(\frac{x\sqrt{\rho} + d/\tau\sigma}{\sqrt{1-\rho}}\right)\right]^p d\Phi(x) \tag{1.3}$$

where $\Phi(\cdot)$ is the standard normal cdf,

$$\tau^2 = \frac{1}{\sigma^2} \mathrm{Var}\{\hat{\alpha}_0 - \hat{\alpha}_i\} = \frac{k(\lambda_0+\lambda_1)}{\lambda_0(\lambda_0+p\lambda_1)} \quad (1 \leq i \leq p), \qquad (1.4)$$

and

$$\rho = \mathrm{Corr}\{\hat{\alpha}_0 - \hat{\alpha}_{i_1}, \hat{\alpha}_0 - \hat{\alpha}_{i_2}\} = \frac{\lambda_1}{\lambda_0+\lambda_1} \quad (i_1 \neq i_2;\ 1 \leq i_1, i_2 \leq p). \quad (1.5)$$

For this problem we seek an optimal design in the class of all admissible BTIB designs.

2. THE CLASS OF ADMISSIBLE DESIGNS

In this section we generalize the definitions of optimal, inadmissible, admissible, and equivalent designs as given in Section 5.1 of [1].

2.1 Optimal and admissible designs

We start by making the following definitions:

Definition 2.1: A BTIB design is said to be b-optimal for given (p,k,b) and specified d/σ if it maximizes the joint confidence coefficient P of (1.3). For given (p,k) and specified d/σ, the b-optimal design which achieves a joint confidence coefficient $\geq 1-\alpha$ with the smallest b-value is said to be optimal for that value of $1-\alpha$.

Definition 2.2: If for given (p,k) we have two BTIB designs D_1 and D_2 with parameters (b_1,τ_1^2,ρ_1) and (b_2,τ_2^2,ρ_2) with $b_1 \leq b_2$, and if for every d and σ, D_1 yields a confidence coefficient at least as large as (larger than) that yielded by D_2 when $b_1 < b_2$ $(b_1 = b_2)$, then we say that D_2 is inadmissible w.r.t. D_1. As in Theorem 5.1 of [1], a

necessary and sufficient condition for D_2 to be inadmissible w.r.t. D_1 is $b_1 \leqq b_2$, $\tau_1^2 \leqq \tau_2^2$ and $\rho_1 \geqq \rho_2$ with at least one inequality strict. If a design is not inadmissible then we say that it is <u>admissible</u>. If $b_1 = b_2$, $\tau_1^2 = \tau_2^2$ and $\rho_1 = \rho_2$, then we say that D_1 and D_2 are <u>equivalent</u>.

If $b_1 = b_2 = b$ (say) then Definition 2.2 reduces to the definition given in Section 5.1 of [1] (since $\tau^2 = \eta^2/kb$ and $\mathrm{Var}\{\hat{\alpha}_0 - \hat{\alpha}_i\} = \sigma^2\eta^2/kb$). Our new definition allows us to compare designs with unequal b's, and to eliminate as inadmissible certain designs with larger b's which would have been admissible under our old definition. Thus, for example, for $(p,k) = (4,3)$ the BTIB designs

$$D_1 = \left\{\begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 0 & 2 & 3 & 3 \\ 2 & 4 & 4 & 3 & 3 & 4 & 4 \end{matrix}\right\}, \quad D_2 = \left\{\begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 4 & 2 & 4 & 4 & 4 \\ 2 & 3 & 3 & 4 & 3 & 4 & 4 & 4 \end{matrix}\right\} \tag{2.1}$$

have $\lambda_0^{(1)} = \lambda_0^{(2)} = 2$, $\lambda_1^{(1)} = \lambda_1^{(2)} = 2$, $b_1 = 7 < b_2 = 8$ where $(\lambda_0^{(i)}, \lambda_1^{(i)}, b_i)$ is associated with D_i $(i = 1,2)$. Hence $\tau_1^2 = \tau_2^2 = 3/5$ and $\rho_1 = \rho_2 = 1/2$, and both D_1 and D_2 yield the <u>same</u> P-value: however, D_1 and D_2 accomplish this with a total of 21 and 24 observations, respectively, and thus D_2 is <u>inadmissible</u> with respect to D_1. Using our former definition, both D_1 and D_2 would have been admissible with respect to designs with $b_1 = 7$ and $b_2 = 8$, respectively, and could not have been compared with each other.

<u>Remark 2.1</u>: Definition 2.2 could be extended to permit comparison of designs having unequal b and/or k values. Thus we would say that if D_1 and D_2 are BTIB designs with parameters $(b_1,k_1,\tau_1^2,\rho_1)$ and

$(b_2,k_2,\tau_2^2,\rho_2)$, respectively, with $N_1 = k_1b_1 \leqq N_2 = k_2b_2$, and if for every d and σ, D_1 yields a confidence coefficient at least as large as (larger than) that yielded by D_2 when $N_1 < N_2$ $(N_1 = N_2)$ then D_2 is inadmissible with respect to D_1; here $k_1,k_2 < p+1$. However, this generalization has not been pursued in the present paper.

Remark 2.2: For given (p,k,b) a design that is admissible in the old sense (and which may maximize (1.3) for some d/σ and that b) may be inadmissible in the new sense, and therefore will not be considered in the search for an optimal design. The deletion of such a design may lead for given (p,k,b) and a particular specified d/σ to a "maximum" confidence coefficient which is smaller than the one that would have been obtained had this design not been deleted; however, in such a case there exists another design for (p,k,b') with $b' < b$ which yields a maximum of (1.3) for that d/σ which is greater than that obtained for b. As an example of this phenomenon for $(p,k) = (4,3)$, the BTIB design

$$\begin{Bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{Bmatrix} \cup \begin{Bmatrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & 4 & 4 & 4 \end{Bmatrix} \cup \begin{Bmatrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & 4 & 4 & 4 \end{Bmatrix} \qquad (2.2)$$

has $\lambda_0 = 2$, $\lambda_1 = 4$, $b = 12$, and is inadmissible (in fact, strongly inadmissible as in Definition 2.4, below) with respect to the BTIB design

$$\begin{Bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 0 & 2 & 3 & 3 \\ 2 & 4 & 4 & 3 & 3 & 4 & 4 \end{Bmatrix} \cup \begin{Bmatrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & 4 & 4 & 4 \end{Bmatrix} \qquad (2.3)$$

which has $\lambda_0 = 2$, $\lambda_1 = 4$, $b' = 11$ since both yield the same confidence coefficient for every d and σ but (2.3) requires a smaller number of blocks. Although (2.2) is inadmissible with respect to (2.3) it actually maximizes P for $0.1 \leq d/\sigma \leq 0.5$ among all (known) BTIB designs with $b = 12$; no loss is sustained by deleting (2.2) from consideration because for $0.1 \leq d/\sigma \leq 0.5$ a BTIB design with $b = 11$ yields a P-value which is at least as large.

2.2 Strongly (S-) inadmissible designs

We repeat here the definition of a generator design given in [1] and [2].

Definition 2.3: A generator design is a BTIB design no proper subset of whose blocks forms a BTIB design.

The candidates for an optimal design for given (p,k) will be all admissible BTIB designs that can be constructed by forming unions of replications of all known generator designs for that given (p,k). It is possible that for some b-values no BTIB design exists; it is also possible that all of the BTIB designs that do exist are inadmissible. (This latter possibility could not occur under our old definition of inadmissibility.) If two or more equivalent admissible BTIB designs exist, then it is necessary to consider only one of them for the purpose of seeking an optimal design. If the union of two or more generator designs yields an equivalent generator design then we will eliminate the latter design from consideration (and thus maintain more flexibility for our construction of designs involving larger numbers of blocks).

This happens, e.g., with $D_1 \cup D_5$ and D_7 in Table 3.1 and $D_4 \cup D_7$ and D_9 in Table 3.2

Certain BTIB designs always yield inadmissible BTIB designs when unions of them are taken with other BTIB designs, and it is desired to eliminate them in our search for an optimal design. For this purpose the concept of strong (S-) inadmissibility (first introduced in Section 3.1 of [2], and generalized here) is needed.

Definition 2.4: If for given (p,k) we have two BTIB designs D_1 and D_2 (not necessarily generator designs), we say that D_2 is S-inadmissible with respect to D_1 if D_2 is inadmissible with respect to D_1 and if for any arbitrary BTIB design D_3 we have that $D_2 \cup D_3$ is inadmissible with respect to $D_1 \cup D_3$; here inadmissibility is in the generalized sense of Definition 2.2. When $b_1 = b_2 = b$ (say) this new definition of S-inadmissibility reduces to our old definition of S-inadmissibility.

Remark 2.3: A sufficient condition for a BTIB design D_2 to be S-inadmissible with respect to a BTIB design D_1 with the same (p,k) is that $b_1 \leq b_2$, $\lambda_0^{(1)} = \lambda_0^{(2)}$, $\lambda_1^{(1)} \geq \lambda_1^{(2)}$ with at least one inequality being strict; here $(\lambda_0^{(i)}, \lambda_1^{(i)}, b_i)$ is associated with D_i $(i = 1,2)$. For $\lambda_1^{(1)} > \lambda_1^{(2)}$ this follows from the fact that for fixed λ_0 the parameter τ^2 of (1.4) is a decreasing function of λ_1, and ρ of (1.5) is an increasing function of λ_1.

2.3 Combination (C-) inadmissible designs

There are certain BTIB designs which are not S-inadmissible but which we can eliminate in our search for an optimal design by using a new concept

which we term combination (C-) inadmissibility. Recall that for a BTIB design D to be not S-inadmissible for given (p,k), D must be such that there does not exist another BTIB design D' such that D is inadmissible wrt D' and for every BTIB D'' we have $D \cup D''$ is inadmissible wrt $D' \cup D''$. Suppose, however, that the BTIB design D is such that for given $D \cup D''$, where D'' is an arbitrary BTIB design, one can always find for that D'' some BTIB design D''' (where $D''' \neq D' \cup D''$ for all D'' for some fixed D') having the property that $D \cup D''$ is inadmissible wrt D'''; we eliminate such D using the concept of C-inadmissibility defined formally below.

Definition 2.5: Suppose that for given (p,k) we have $n \geq 2$ BTIB generator designs D_i $(1 \leq i \leq n)$ no two of which are equivalent, and no one of which is equivalent to the union of two or more generator designs. Suppose further that no D_i $(1 \leq i \leq n)$ is S-inadmissible. Consider a design $D = \bigcup_{i=1}^{n} f_i D_i$ and an arbitrary design $D'' = \bigcup_{i=1}^{n} g_i D_i$. If given $D \cup D''$ one can find a design $D''' = \bigcup_{i=1}^{n} h_i D_i$ such that $D \cup D''$ is inadmissible wrt D''', then we say that D is C-inadmissible wrt the set $\{D_1, D_2, \ldots, D_n\}$.

Remark 2.4: If the above definition is satisfied except for some finite number of unions $D \cup D''$ then we shall refer to D as being C-inadmissible with the exceptions noted. Thus, in particular, if D'' is an empty design (i.e., all $g_i = 0$), then D itself can be admissible.

Remark 2.5: It is to be noted that C-inadmissibility is defined with respect to the set of designs $\{D_1, \ldots, D_n\}$. A design may be C-inadmissible

wrt $\{D_1,\ldots,D_n\}$ but <u>not</u> C-inadmissible wrt $\{D_1,\ldots,D_n,D_{n+1}\}$ where this new set containing the one additional BTIB generator design is such that no two of the designs are equivalent, no one of the designs is equivalent to the union of two or more generator designs, and no D_i $(1 \leq i \leq n+1)$ is S-inadmissible. Thus D cannot be deleted unless it is known that the set contains <u>all</u> nonequivalent and non S-inadmissible designs. This is in contrast to S-inadmissibility wherein an S-inadmissible design can always be deleted without any loss.

<u>Remark 2.6</u>: If a design $D_i \in \{D_1,\ldots,D_n\}$ is C-inadmissible with respect to that set then we shall say that D_i is C-inadmissible with respect to $\{D_j\ (j \neq i,\ 1 \leq j \leq n)\}$. The reason for doing so is that D_i is C-inadmissible implies that for given (p,k) and an arbitrary BTIB design D there exists a frequency vector $(f_1,\ldots,f_{i-1},0,f_{i+1},\ldots,f_n)$ such that $D_i \cup D$ is inadmissible with respect to $\bigcup_{j=1}^{n} f_j D_j$.

We are now led to our final definition.

<u>Definition 2.6</u>: If the set $\{D_1,\ldots,D_n\}$ contains <u>all</u> generator designs for given (p,k), and if $\{D_{i_1},\ldots,D_{i_m}\}$ with $m \leq n$ is the subset which contains all generator designs which are nonequivalent and non S-inadmissible then the latter set will be referred to as the <u>minimal complete class</u> of generator designs for given (p,k).

<u>Remark 2.7</u>: We conjecture that the set $\{D_1,D_2,D_3,D_4,D_5\}$ where the D_i $(1 \leq i \leq 5)$ are defined in Table 3.1 is the minimal complete class for $(p = 4,\ k = 3)$ and that the set $\{D_1,D_2,D_3,D_4,D_5,D_6\}$ where the D_i $(1 \leq i \leq 6)$ are defined in Table 3.2 is the minimal complete class for

($p = 5$, $k = 3$). With respect to these sets the concept of C-inadmissibility can then be used to eliminate D_1 for ($p = 4$, $k = 3$) and $b \neq 4$, and D_1 for ($p = 5$, $k = 3$) and $b \neq 5,15$.

A C-inadmissible design is not as easy to identify as an S-inadmissible design. It is necessary to examine every different elementary combination of generator designs, and in some cases higher order combinations, and show that each such combination leads to inadmissible designs. Examples of designs which are C-inadmissible with respect to all generator designs known to us are given in the next section; proofs of their C-inadmissibility are given in the Appendix.

With these new definitions of inadmissibility the class of BTIB designs which are eligible to be considered as contenders for optimal designs can be greatly reduced, and the optimization problem as posed in (3.1) and (3.2) of [2] becomes much easier to solve. Only one of each of the equivalent designs need be considered, and all of the S-inadmissible designs can be eliminated. The C-inadmissible designs can also be eliminated except possibly for certain b-values.

3. RESULTS FOR $p = 4$, $k = 3$ AND $p = 5$, $k = 3$

3.1 Lists of generator designs

The generator designs that we have constructed (by the methods described in Section 3.2 of [1], or by other methods) are listed for $p = 4$, $k = 3$ and $p = 5$, $k = 3$ in Tables 3.1 and 3.2, respectively. Generator designs which are equivalent are given only one label as

with the designs labeled D_3 in Table 3.1. We have not exhibited equivalent designs which differ only trivially from those given in the tables, e.g.,

$$\left\{\begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{matrix}\right\}, \left\{\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{matrix}\right\}, \left\{\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{matrix}\right\}$$

are equivalent to but differ trivially from D_1 in Table 3.1. Equivalent designs can be used interchangeably without affecting their statistical properties.

For the generator designs listed in Table 3.1 we note that: a) D_6 is S-inadmissible wrt D_2, b) D_7 is equivalent to $D_1 \cup D_5$ and is S-inadmissible wrt D_3, c) D_8 is S-inadmissible wrt $3D_5$, and d) D_1 is C-inadmissible wrt $\{D_2,D_3,D_4,D_5\}$ for all $b \neq 4$; this latter result is proved in the Appendix. Thus for $p = 4$, $k = 3$ we employ D_1 for $b = 4$, and it suffices to consider unions of replications of D_2, D_3, D_4, D_5 for $b > 4$ when seeking the optimal design for a specified d/σ.

For the generator designs listed in Table 3.2 we note that: a) D_8 is S-inadmissible wrt D_4, b) D_9 is equivalent to $D_4 \cup D_7$ and is S-inadmissible wrt $D_2 \cup D_3$, c) D_{10} is S-inadmissible wrt $2D_6$ and $2D_7$, d) D_7 is C-inadmissible wrt $\{D_1,D_2,D_3,D_4,D_5,D_6\}$, and since it contains no control treatments (and thus is unimplementable by itself) it should <u>never</u> be used, and e) D_1 is C-inadmissible wrt $\{D_2,D_3,D_4,D_5,D_6\}$ except for $D_1 \cup D_4$. These latter results are proved in the Appendix. Thus for $p = 5$, $k = 3$ we employ D_1 for $b = 5$ and $D_1 \cup D_4$ for $b = 15$,

Table 3.1

Generator Designs for p = 4, k = 3

Label	Design	b_i	$\lambda_0^{(i)}$	$\lambda_1^{(i)}$
D_1	$\left\{\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{matrix}\right\}$	4	2	0
D_2	$\left\{\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 4 & 3 & 4 & 4 \end{matrix}\right\}$	6	3	1
D_3	$\left\{\begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 0 & 2 & 3 & 3 \\ 2 & 4 & 4 & 3 & 3 & 4 & 4 \end{matrix}\right\}, \left\{\begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 3 \\ 1 & 1 & 2 & 2 & 2 & 2 & 4 \\ 3 & 4 & 3 & 4 & 3 & 4 & 4 \end{matrix}\right\}$	7	2	2
D_4	$\left\{\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 2 & 2 & 0 & 0 & 3 & 3 \\ 2 & 2 & 3 & 4 & 3 & 4 & 3 & 4 & 4 & 4 \end{matrix}\right\}$	10	4	2
D_5	$\left\{\begin{matrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & 4 & 4 & 4 \end{matrix}\right\}$	4	0	2
D_6	$\left\{\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 0 & 0 & 0 & 2 \\ 3 & 3 & 4 & 1 & 2 & 4 & 4 \end{matrix}\right\}$	7	3	1
D_7	$\left\{\begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 4 & 2 & 4 & 4 & 4 \\ 2 & 3 & 3 & 4 & 3 & 4 & 4 & 4 \end{matrix}\right\}$	8	2	2
D_8	$\left\{\begin{matrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 \\ 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 \\ 2 & 3 & 4 & 1 & 3 & 4 & 1 & 2 & 4 & 1 & 2 & 3 \end{matrix}\right\}$	12	0	4

Table 3.2

Generator Designs for p = 5, k = 3

Label	Design	b_i	$\lambda_0^{(i)}$	$\lambda_1^{(i)}$
D_1	$\left\{\begin{array}{ccccc} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 1&2&3&4&5 \end{array}\right\}$	5	2	0
D_2	$\left\{\begin{array}{ccccccc} 0&0&0&0&0&1&2 \\ 1&1&3&4&0&2&3 \\ 3&5&4&5&2&4&5 \end{array}\right\}$	7	2	1
D_3	$\left\{\begin{array}{cccccccccc} 0&0&0&0&0&1&1&1&2&2 \\ 1&1&2&3&4&2&3&4&3&3 \\ 2&3&4&5&5&5&4&5&4&5 \end{array}\right\}$	10	2	2
D_4	$\left\{\begin{array}{cccccccccc} 0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&2&2&2&3&3&4 \\ 2&3&4&5&3&4&5&4&5&5 \end{array}\right\}$	10	4	1
D_5	$\left\{\begin{array}{cccccccccccccc} 0&0&0&0&0&1&1&1&1&1&2&2&2&3 \\ 1&1&3&0&0&2&2&2&3&4&3&3&4&4 \\ 3&4&4&2&5&3&4&5&5&5&4&5&5&5 \end{array}\right\}$	14	2	3
D_6	$\left\{\begin{array}{cccccccccc} 1&1&1&1&1&1&2&2&2&3 \\ 2&2&2&3&3&4&3&3&4&4 \\ 3&4&5&4&5&5&4&5&5&5 \end{array}\right\}$	10	0	3
D_7	$\left\{\begin{array}{cccccccc} 1&1&1&2&1&2&3&4 \\ 2&2&3&3&5&5&5&5 \\ 3&4&4&4&5&5&5&5 \end{array}\right\}$	8	0	2
D_8	$\left\{\begin{array}{ccccccccccc} 0&0&0&0&0&0&0&0&0&0&1 \\ 1&1&2&2&2&3&4&0&0&0&3 \\ 2&5&3&4&5&5&5&1&3&4&4 \end{array}\right\}$	11	4	1
D_9	$\left\{\begin{array}{cccccccccccccccccc} 0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&2&2&3 \\ 1&1&2&2&3&4&0&0&0&0&2&2&2&3&4&3&3&4 \\ 3&4&4&5&4&5&1&2&3&5&3&4&5&5&5&4&5&5 \end{array}\right\}$	18	4	3
D_{10}	$\left\{\begin{array}{cccccccccccccccccccc} 1&1&1&1&2&2&2&2&3&3&3&3&4&4&4&4&5&5&5&5 \\ 1&1&1&1&2&2&2&2&3&3&3&3&4&4&4&4&5&5&5&5 \\ 2&3&4&5&1&3&4&5&1&2&4&5&1&2&3&5&1&2&3&4 \end{array}\right\}$	20	0	4

and it suffices to consider unions of replications of D_2, D_3, D_4, D_5, D_6 for $b \neq 5,15$ when seeking the optimal design for a specified d/σ.

3.2 Catalogs of admissible designs

Catalogs of admissible designs have been prepared based on the set of admissible generator designs given for $p = 4$, $k = 3$ in Table 3.1 and for $p = 5$, $k = 3$ in Table 3.2. These catalogs are given in Table 3.3 for $p = 4$, $k = 3$; $b = 6,7,10(1)46$ and in Table 3.4 for $p = 5$, $k = 3$; $b = 5,7,10,14,15,17,20,24,27,30,34,37,40,44,47,50,54$. For $p = 4$, $k = 3$ and $p = 5$, $k = 3$ BTIB designs do not exist for $b = 5,9$ and $6,9$, respectively. For other b-values for which no design is listed, i.e., $b = 8$ for $p = 4$, $k = 3$ and $b = 8,11,12,13,16,18,19,21,22,23,26,28,$ $29,31,32,33,35,36,38,39,41,42,43,45,46,48,49,51,52,53$ for $p = 5$, $k = 3$, BTIB designs (which are S-inadmissible) do exist, but BTIB designs which involve smaller b-values are preferable. It is also interesting to note that for $p = 4$, $k = 3$ only one admissible BTIB design exists for $b = 4,6,7,11,12,13,15,17$ and 21, and thus for these numbers of blocks the given design is optimal for all d and σ both for one-sided and two-sided intervals; for $p = 5$, $k = 3$ the same is true for $b = 5$, 7, and 15.

Remark 3.1: For $p = 5$, $k = 3$ we note that the designs $D_4 \cup D_5 \cup fD_6$ and $2D_2 \cup D_3 \cup fD_6$ are equivalent and admissible for $f = 0,1,2,\ldots$. If we choose to employ only $2D_2 \cup D_3 \cup fD_6$ for $f = 0,1,2,\ldots$ then D_5 will be used only in the combination $D_5 \cup gD_6$ for $g = 0,1,2,\ldots$ for which it is admissible; for $b = 10t + 4$ $(t = 1,2,\ldots)$ this admissible

Table 3.3

Catalog of admissible designs[1/] for $p = 4$, $k = 3$

No. of blocks (b)	D_1 $b_1 = 4$ $\lambda_0^{(1)} = 2$ $\lambda_1^{(1)} = 0$	D_2 $b_2 = 6$ $\lambda_0^{(2)} = 3$ $\lambda_1^{(2)} = 1$	D_3 $b_3 = 7$ $\lambda_0^{(3)} = 2$ $\lambda_1^{(3)} = 2$	D_4 $b_4 = 10$ $\lambda_0^{(4)} = 4$ $\lambda_1^{(4)} = 2$	D_5 $b_5 = 4$ $\lambda_0^{(5)} = 0$ $\lambda_1^{(5)} = 2$	λ_0	λ_1	τ^2	ρ
4	1	0	0	0	0	2	0	1.5000	0.000
6	0	1	0	0	0	3	1	0.5714	0.250
7	0	0	1	0	0	2	2	0.6000	0.500
10	0	1	0	0	1	3	3	0.4000	0.500
	0	0	0	1	0	4	2	0.3750	0.333
11	0	0	1	0	1	2	4	0.5000	0.667
12	0	2	0	0	0	6	2	0.2857	0.250
13	0	1	1	0	0	5	3	0.2824	0.375
14	0	1	0	0	2	3	5	0.3478	0.625
	0	0	0	1	1	4	4	0.3000	0.500
15	0	0	1	0	2	2	6	0.4615	0.750
16	0	2	0	0	1	6	4	0.2273	0.400
	0	1	0	1	0	7	3	0.2256	0.300
17	0	1	1	0	1	5	5	0.2400	0.500
18	0	1	0	0	3	3	7	0.3226	0.700
	0	0	2	0	1	4	6	0.2679	0.600
	0	3	0	0	0	9	3	0.1905	0.250
19	0	0	1	0	3	2	8	0.4412	0.800
	0	2	1	0	0	8	4	0.1875	0.333
20	0	2	0	0	2	6	6	0.2000	0.500
	0	1	0	1	1	7	5	0.1905	0.417

[1/] For each number of blocks, the number under D_i $(1 \leq i \leq 5)$ in the body of the table is the frequency f_i with which D_i appears in the design $D = \bigcup_{i=1}^{5} f_i D_i$.

Table 3.3 (continued)

No. of blocks (b)	D_1 $b_1 = 4$ $\lambda_0^{(1)} = 2$ $\lambda_1^{(1)} = 0$	D_2 $b_2 = 6$ $\lambda_0^{(2)} = 3$ $\lambda_1^{(2)} = 1$	D_3 $b_3 = 7$ $\lambda_0^{(3)} = 2$ $\lambda_1^{(3)} = 2$	D_4 $b_4 = 10$ $\lambda_0^{(4)} = 4$ $\lambda_1^{(4)} = 2$	D_5 $b_5 = 4$ $\lambda_0^{(5)} = 0$ $\lambda_1^{(5)} = 2$	λ_0	λ_1	τ^2	ρ
21	0	1	1	0	2	5	7	0.2182	0.583
22	0	1	0	0	4	3	9	0.3077	0.750
	0	0	2	0	2	4	8	0.2500	0.667
	0	3	0	0	1	9	5	0.1609	0.357
23	0	0	1	0	4	2	10	0.4296	0.833
	0	2	1	0	1	8	6	0.1641	0.429
24	0	2	0	0	3	6	8	0.1842	0.571
	0	1	0	1	2	7	7	0.1714	0.500
	0	4	0	0	0	12	4	0.1429	0.250
25	0	1	1	0	3	5	9	0.2049	0.643
	0	3	1	0	0	11	5	0.1408	0.313
26	0	1	0	0	5	3	11	0.2979	0.786
	0	0	2	0	3	4	10	0.2356	0.714
	0	3	0	0	2	9	7	0.1441	0.438
	0	2	0	1	1	10	6	0.1412	0.375
27	0	0	1	0	5	2	12	0.4200	0.857
	0	2	1	0	2	8	8	0.1500	0.500
28	0	2	0	0	4	6	10	0.1739	0.625
	0	1	0	1	3	7	9	0.1595	0.563
	0	4	0	0	1	12	6	0.1250	0.333
29	0	1	1	0	4	5	11	0.1959	0.688
	0	3	1	0	1	11	7	0.1259	0.389
30	0	1	0	0	6	3	13	0.2909	0.813
	0	0	2	0	4	4	12	0.2308	0.750
	0	3	0	0	3	9	9	0.1333	0.500
	0	2	0	1	2	10	8	0.1296	0.444
	0	5	0	0	0	15	5	0.1143	0.250
31	0	0	1	0	6	2	14	0.4133	0.875
	0	2	1	0	3	8	10	0.1406	0.556
	0	4	1	0	0	14	6	0.1128	0.300

Table 3.3 (continued)

No. of blocks (b)	D_1 $b_1 = 4$ $\lambda_0^{(1)} = 2$ $\lambda_1^{(1)} = 0$	D_2 $b_2 = 6$ $\lambda_0^{(2)} = 3$ $\lambda_1^{(2)} = 1$	D_3 $b_3 = 7$ $\lambda_0^{(3)} = 2$ $\lambda_1^{(3)} = 2$	D_4 $b_4 = 10$ $\lambda_0^{(4)} = 4$ $\lambda_1^{(4)} = 2$	D_5 $b_5 = 4$ $\lambda_0^{(5)} = 0$ $\lambda_1^{(5)} = 2$	λ_0	λ_1	τ^2	ρ
32	0	2	0	0	5	6	12	0.1667	0.667
	0	1	0	1	4	7	11	0.1513	0.611
	0	4	0	0	2	12	8	0.1136	0.400
	0	3	0	1	1	13	7	0.1126	0.350
33	0	1	1	0	5	5	13	0.1895	0.722
	0	3	1	0	2	11	9	0.1161	0.450
34	0	1	0	0	7	3	15	0.2857	0.833
	0	0	2	0	5	4	14	0.2250	0.778
	0	3	0	0	4	9	11	0.1258	0.550
	0	2	0	1	3	10	10	0.1200	0.500
	0	5	0	0	1	15	7	0.1023	0.319
35	0	0	1	0	7	2	16	0.4091	0.889
	0	2	1	0	4	8	12	0.1339	0.600
	0	4	1	0	1	14	8	0.1025	0.364
36	0	2	0	0	6	6	14	0.1613	0.700
	0	1	0	1	5	7	13	0.1453	0.650
	0	4	0	0	3	12	10	0.1058	0.455
	0	3	0	1	2	13	9	0.1036	0.409
	0	6	0	0	0	18	6	0.0952	0.250
37	0	1	1	0	6	5	15	0.1846	0.750
	0	3	1	0	3	11	11	0.1091	0.500
	0	5	1	0	0	17	7	0.0941	0.292
38	0	1	0	0	8	3	17	0.2817	0.850
	0	0	2	0	6	4	16	0.2206	0.800
	0	3	0	0	5	9	13	0.1202	0.591
	0	2	0	1	4	10	12	0.1138	0.545
	0	5	0	0	2	15	9	0.0941	0.375
	0	4	0	1	1	16	8	0.0938	0.333
39	0	0	1	0	8	2	18	0.4054	0.900
	0	2	1	0	5	8	14	0.1289	0.635
	0	4	1	0	2	14	10	0.0952	0.417

Table 3.3 (continued)

No. of blocks (b)	D_1: $b_1 = 4$, $\lambda_0^{(1)} = 2$, $\lambda_1^{(1)} = 0$	D_2: $b_2 = 6$, $\lambda_0^{(2)} = 3$, $\lambda_1^{(2)} = 1$	D_3: $b_3 = 7$, $\lambda_0^{(3)} = 2$, $\lambda_1^{(3)} = 2$	D_4: $b_4 = 10$, $\lambda_0^{(4)} = 4$, $\lambda_1^{(4)} = 2$	D_5: $b_5 = 4$, $\lambda_0^{(5)} = 0$, $\lambda_1^{(5)} = 2$	λ_0	λ_1	τ^2	ρ
40	0	2	0	0	7	6	16	0.1571	0.727
	0	1	0	1	6	7	15	0.1407	0.682
	0	4	0	0	4	12	12	0.1000	0.500
	0	3	0	1	3	13	11	0.0972	0.458
	0	6	0	0	1	18	8	0.0867	0.308
41	0	1	1	0	7	5	17	0.1308	0.773
	0	3	1	0	4	11	13	0.1039	0.542
	0	5	1	0	1	17	9	0.0866	0.346
42	0	1	0	0	9	3	19	0.2785	0.864
	0	0	2	0	7	4	18	0.2171	0.818
	0	3	0	0	6	9	15	0.1159	0.625
	0	2	0	1	5	10	14	0.1091	0.583
	0	5	0	0	3	15	11	0.0881	0.423
	0	4	0	1	2	16	10	0.0871	0.385
	0	7	0	0	0	21	7	0.0816	0.250
43	0	0	1	0	9	2	20	0.4024	0.909
	0	2	1	0	3	8	16	0.1250	0.667
	0	4	1	0	3	14	12	0.0899	0.462
	0	6	1	0	0	20	8	0.0808	0.286
44	0	2	0	0	8	6	18	0.1538	0.750
	0	1	0	1	7	7	17	0.1371	0.708
	0	4	0	0	5	12	14	0.0956	0.538
	0	3	0	1	4	13	13	0.0923	0.500
	0	6	0	0	2	18	10	0.0805	0.357
	0	5	0	1	1	19	9	0.0804	0.321
45	0	1	1	0	8	5	19	0.1778	0.792
	0	3	1	0	5	11	15	0.0999	0.577
	0	5	1	0	2	17	11	0.0810	0.393
46	0	1	0	0	10	3	21	0.2759	0.875
	0	0	2	0	8	4	20	0.2143	0.833
	0	3	0	0	7	9	17	0.1126	0.654
	0	2	0	1	6	10	16	0.1054	0.615
	0	5	0	0	4	15	13	0.0836	0.464
	0	4	0	1	3	16	12	0.0820	0.429
	0	7	0	0	1	21	9	0.0752	0.300

Table 3.4

Catalog of admissible designs[1/] for p = 5, k = 3

No. of blocks (b)	D_1 $b_1 = 5$ $\lambda_0^{(1)} = 2$ $\lambda_1^{(1)} = 0$	D_2 $b_2 = 7$ $\lambda_0^{(2)} = 2$ $\lambda_1^{(2)} = 1$	D_3 $b_3 = 10$ $\lambda_0^{(3)} = 2$ $\lambda_1^{(3)} = 2$	D_4 $b_4 = 10$ $\lambda_0^{(4)} = 4$ $\lambda_1^{(4)} = 1$	D_5 $b_5 = 14$ $\lambda_0^{(5)} = 2$ $\lambda_1^{(5)} = 3$	D_6 $b_6 = 10$ $\lambda_0^{(6)} = 0$ $\lambda_1^{(6)} = 3$	λ_0	λ_1	τ^2	ρ
5	1	0	0	0	0	0	2	0	1.5000	0.000
7	0	1	0	0	0	0	2	1	0.6429	0.333
10	0	0	1	0	0	0	2	2	0.5000	0.500
	0	0	0	1	0	0	4	1	0.4167	0.200
14	0	0	0	0	1	0	2	3	0.4412	0.600
	0	2	0	0	0	0	4	2	0.3214	0.333
15	1	0	0	1	0	0	6	1	0.3182	0.143
17	0	1	0	0	0	1	2	4	0.4091	0.667
	0	1	1	0	0	0	4	3	0.2763	0.429
	0	1	0	1	0	0	6	2	0.2500	0.250
20	0	0	1	0	0	1	2	5	0.3889	0.714
	0	0	0	1	0	1	4	4	0.2500	0.500
	0	0	1	1	0	0	6	3	0.2143	0.333
	0	0	0	2	0	0	8	2	0.2083	0.200
24	0	0	0	0	1	1	2	6	0.3750	0.750
	0	2	0	0	0	1	4	5	0.2328	0.556
	0	2	1	0	0	0	6	4	0.1923	0.400
	0	2	0	1	0	0	8	3	0.1793	0.273
27	0	1	0	0	0	2	2	7	0.3649	0.778
	0	1	1	0	0	1	4	6	0.2206	0.600
	0	1	0	1	0	1	6	5	0.1774	0.455
	0	1	1	1	0	0	8	4	0.1607	0.333
	0	1	0	2	0	0	10	3	0.1560	0.231

[1/] For each number of blocks, the number under D_i $(1 \leq i \leq 6)$ in the body of the table is the frequency f_i with which D_i appears in the design $D = \bigcup_{i=1}^{6} f_i D_i$.

Table 3.4 (continued)

No. of blocks (b)	D_1 $b_1 = 5$ $\lambda_0^{(1)} = 2$ $\lambda_1^{(1)} = 0$	D_2 $b_2 = 7$ $\lambda_0^{(2)} = 2$ $\lambda_1^{(2)} = 1$	D_3 $b_3 = 10$ $\lambda_0^{(3)} = 2$ $\lambda_1^{(3)} = 2$	D_4 $b_4 = 10$ $\lambda_0^{(4)} = 4$ $\lambda_1^{(4)} = 1$	D_5 $b_5 = 14$ $\lambda_0^{(5)} = 2$ $\lambda_1^{(5)} = 3$	D_6 $b_6 = 10$ $\lambda_0^{(6)} = 0$ $\lambda_1^{(6)} = 3$	λ_0	λ_1	τ^2	ρ
30	0	0	1	0	0	2	2	8	0.3571	0.800
	0	0	0	1	0	2	4	7	0.2115	0.636
	0	0	1	1	0	1	6	6	0.1667	0.500
	0	0	0	2	0	1	8	5	0.1477	0.385
	0	0	1	2	0	0	10	4	0.1400	0.286
	0	0	0	3	0	0	12	3	0.1389	0.200
34	0	0	0	0	1	2	2	9	0.3511	0.818
	0	2	0	0	0	2	4	8	0.2045	0.667
	0	2	1	0	0	1	6	7	0.1585	0.538
	0	2	0	1	0	1	8	6	0.1382	0.429
	0	2	1	1	0	0	10	5	0.1286	0.333
	0	2	0	2	0	0	12	4	0.1250	0.250
37	0	1	0	0	0	3	2	10	0.3462	0.833
	0	1	1	0	0	2	4	9	0.1990	0.692
	0	1	0	1	0	2	6	8	0.1522	0.571
	0	1	1	1	0	1	8	7	0.1308	0.467
	0	1	0	2	0	1	10	6	0.1200	0.375
	0	1	1	2	0	0	12	5	0.1149	0.294
	0	1	0	3	0	0	14	4	0.1134	0.222
40	0	0	1	0	0	3	2	11	0.3421	0.846
	0	0	0	1	0	3	4	10	0.1944	0.714
	0	0	1	1	0	2	6	9	0.1471	0.600
	0	0	2	1	0	1	8	8	0.1250	0.500
	0	0	1	2	0	1	10	7	0.1133	0.412
	0	0	0	3	0	1	12	6	0.1071	0.333
	0	0	1	3	0	0	14	5	0.1044	0.263
	0	0	0	4	0	0	16	4	0.1042	0.200
44	0	0	0	0	1	3	2	12	0.3387	0.857
	0	2	0	0	0	3	4	11	0.1907	0.733
	0	2	1	0	0	2	6	10	0.1429	0.625
	0	2	0	1	0	2	8	9	0.1203	0.529
	0	2	1	1	0	1	10	8	0.1080	0.444
	0	2	0	2	0	1	12	7	0.1011	0.368
	0	2	1	2	0	0	14	6	0.0974	0.300
	0	2	0	3	0	0	16	5	0.0960	0.238

Table 3.4 (continued)

No. of blocks (b)	D_1 $b_1 = 5$ $\lambda_0^{(1)} = 2$ $\lambda_1^{(1)} = 0$	D_2 $b_2 = 7$ $\lambda_0^{(2)} = 2$ $\lambda_1^{(2)} = 1$	D_3 $b_3 = 10$ $\lambda_0^{(3)} = 2$ $\lambda_1^{(3)} = 2$	D_4 $b_4 = 10$ $\lambda_0^{(4)} = 4$ $\lambda_1^{(4)} = 1$	D_5 $b_5 = 14$ $\lambda_0^{(5)} = 2$ $\lambda_1^{(5)} = 3$	D_6 $b_6 = 10$ $\lambda_0^{(6)} = 0$ $\lambda_1^{(6)} = 3$	λ_0	λ_1	τ^2	ρ
47	0	1	0	0	0	4	2	13	0.3358	0.867
	0	1	1	0	0	3	4	12	0.1875	0.750
	0	1	0	1	0	3	6	11	0.1393	0.647
	0	1	1	1	0	2	8	10	0.1164	0.556
	0	1	0	2	0	2	10	9	0.1036	0.474
	0	1	1	2	0	1	12	8	0.0962	0.400
	0	1	0	3	0	1	14	7	0.0918	0.333
	0	1	1	3	0	0	16	6	0.0897	0.273
	0	1	0	4	0	0	18	5	0.0891	0.217
50	0	0	1	0	0	4	2	14	0.3333	0.875
	0	0	0	1	0	4	4	13	0.1848	0.765
	0	0	1	1	0	3	6	12	0.1364	0.667
	0	0	2	1	0	2	8	11	0.1131	0.579
	0	0	1	2	0	2	10	10	0.1000	0.500
	0	0	2	2	0	1	12	9	0.0921	0.429
	0	0	1	3	0	1	14	8	0.0873	0.364
	0	0	0	4	0	1	16	7	0.0846	0.304
	0	0	1	4	0	0	18	6	0.0833	0.250
54	0	0	0	0	1	4	2	15	0.3312	0.882
	0	2	0	0	0	4	4	14	0.1824	0.778
	0	2	1	0	0	3	6	13	0.1338	0.684
	0	2	0	1	0	3	8	12	0.1103	0.600
	0	2	1	1	0	2	10	11	0.0969	0.524
	0	2	0	2	0	2	12	10	0.0887	0.455
	0	2	1	2	0	1	14	9	0.0835	0.391
	0	2	0	3	0	1	16	8	0.0804	0.333
	0	2	1	3	0	0	18	7	0.0786	0.280
	0	2	0	4	0	0	20	6	0.0780	0.231

design is then associated with the largest ρ-value for that b, and thus is always optimal if d/σ is sufficiently small.

3.3 Tables of optimal designs

Optimal designs for $p = 4$, $k = 3$ are given in Table 3.5 for $d/\sigma = 0.1(0.1)1.0$ and $b = 4 - 47$. If no design is given for a particular $(b,d/\sigma)$-combination, then either no BTIB design exists for that b-value or one or more BTIB design (all of which are inadmissible) do exist, but BTIB designs which involve smaller b-values are preferable. Similarly, optimal designs for $p = 5$, $k = 3$ are given in Table 3.6 for $d/\sigma = 0.1(0.1)1.0$ and $b = 5 - 67$.

Remark 3.2: For $p = 5$, $k = 3$ we have checked for all $b \leq 54$ that every admissible design in Table 3.4 is optimal for some d/σ; whether or not this phenomenon is true for all b for $p = 5$, $k = 3$ is unknown to us. On the other hand, we know that this phenomenon is not true for all b for $p = 4$, $k = 3$: in fact for $p = 4$, $k = 3$ every admissible design in Table 3.3 is optimal for some d/σ for $b \leq 26$; but, e.g., for $b = 28,30,32$ the admissible designs which are not optimal for any d/σ are given in Table 3.7.

Table 3.5

Optimal Designs[1/] and Associated Confidence Coefficient (P) as a Function of

b and d/σ for p = 4, k = 3

No. of blocks (b)	d/σ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
4	1,0,0 0,0 0.0804	1,0,0 0,0 0.1018	1,0,0 0,0 0.1268	1,0,0 0,0 0.1556	1,0,0 0,0 0.1880	1,0,0 0,0 0.2239	1,0,0 0,0 0.2631	1,0,0 0,0 0.3051	1,0,0 0,0 0.3493	1,0,0 0,0 0.3952
6	0,1,0 0,0 0.1666	0,1,0 0,0 0.2139	0,1,0 0,0 0.2678	0,1,0 0,0 0.3275	0,1,0 0,0 0.3915	0,1,0 0,0 0.4582	0,1,0 0,0 0.5257	0,1,0 0,0 0.5919	0,1,0 0,0 0.6552	0,1,0 0,0 0.7140
7	0,0,1 0,0 0.2451	0,0,1 0,0 0.2950	0,0,1 0,0 0.3490	0,0,1 0,0 0.4061	0,0,1 0,0 0.4652	0,0,1 0,0 0.5249	0,0,1 0,0 0.5839	0,0,1 0,0 0.6409	0,0,1 0,0 0.6948	0,0,1 0,0 0.7447
10	0.1,0 0,1 0.2559	0,1,0 0,1 0.3188	0,1,0 0,1 0.3872	0,1,0 0,1 0.4592	0,1,0 0,1 0.5323	0,1,0 0,1 0.6041	0,1,0 0,1 0.6723	0,1,0 0,1 0.7350	0,1,0 0,1 0.7907	0,1,0 0,1 0.8388
11	0,0,1 0,1 0.3105	0,0,1 0,1 0.3675	0,0,1 0,1 0.4274	0,0,1 0,1 0.4890	0,0,1 0,1 0.5508	0,0,1 0,1 0.6113				
12						0,2,0 0,0 0.6231	0,2,0 0,0 0.7083	0,2,0 0,0 0.7825	0,2,0 0,0 0.8439	0,2,0 0,0 0.8921

[1/] The "matrix" in each cell is $\begin{Bmatrix} \hat{f}_1, \hat{f}_2, \hat{f}_3 \\ \hat{f}_4, \hat{f}_5 \end{Bmatrix}$ where $\hat{D} = \bigcup_{i=1}^{5} \hat{f}_i D_i$ with $b = \sum_{i=1}^{5} \hat{f}_i b_i$ is the optimal design for the given value of b and d/σ.

Table 3.5 (continued)

No. of blocks	d/σ									
(b)	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
13					0,1,1 0,0 0.5665	0,1,1 0,0 0.6533	0,1,1 0,0 0.7323	0,1,1 0,0 0.8005	0,1,1 0,0 0.8567	0,1,1 0,0 0.9009
14		0,1,0 0,2 0.3746	0,1,0 0,2 0.4480	0,1,0 0,2 0.5231	0,1,0 0,2 0.5973	0,1,0 0,2 0.6680	0,0,0 1,1 0.7399	0,0,0 1,1 0.8029		
15	0,0,1 0,2 0.3477	0,0,1 0,2 0.4076	0,0,1 0,2 0.4697	0,0,1 0,2 0.5325						
16					0,2,0 0,1 0.6230	0,2,0 0,1 0.7133	0,2,0 0,1 0.7913	0,2,0 0,1 0.8548	0,2,0 0,1 0.9034	0,2,0 0,1 0.9387
17				0,1,1 0,1 0.5442	0,1,1 0,1 0.6357	0,1,1 0,1 0.7197	0,1,1 0,1 0.7927			
18		0,1,0 0,3 0.4102	0,1,0 0,3 0.4858	0,1,0 0,3 0.5619	0,0,2 0,1 0.6398	0,3,0 0,0 0.7356	0,3,0 0,0 0.8193	0,3,0 0,0 0.8834	0,3,0 0,0 0.9289	0,3,0 0,0 0.9590
19	0,0,1 0,3 0.3724	0,0,1 0,3 0.4339	0,0,1 0,3 0.4970		0,2,1 0,0 0.6557	0,2,1 0,0 0.7521	0,2,1 0,0 0.8311	0,2,1 0,0 0.8912	0,2,1 0,0 0.9338	0,2,1 0,0 0.9619

Table 3.5 (continued)

No. of blocks	d/σ									
(b)	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
20				0,2,0 0,2 0.5797	0,2,0 0,2 0.6769	0,2,0 0,2 0.7630	0,1,0 1,1 0.8358	0,1,0 1,1 0.8932	0,1,0 1,1 0.9342	
21				0,1,1 0,2 0.5878	0,1,1 0,2 0.6781					
22		0,1,0 0,4 0.4353	0,1,0 0,4 0.5121	0,0,2 0,2 0.5908	0,3,0 0,1 0.7003	0,3,0 0,1 0.7950	0,3,0 0,1 0.8686	0,3,0 0,1 0.9211	0,3,0 0,1 0.9556	0,3,0 0,1 0.9766
23	0,0,1 0,4 0.3903	0,0,1 0,4 0.4527	0,0,1 0,4 0.5163	0,2,1 0,1 0.6024	0,2,1 0,1 0.7091	0,2,1 0,1 0.7995	0,2,1 0,1 0.8701			
24				0,2,0 0,3 0.6171	0,4,0 0,0 0.7140	0,4,0 0,0 0.8140	0,4,0 0,0 0.8878	0,4,0 0,0 0.9373	0,4,0 0,0 0.9674	0,4,0 0,0 0.9843
25			0,1,1 0,3 0.5227	0,1,1 0,3 0.6183	0,3,1 0,0 0.7280	0,3,1 0,0 0.8237	0,3,1 0,0 0.8941	0,3,1 0,0 0.9410	0,3,1 0,0 0.9695	0,3,1 0,0 0.9854
26		0,1,0 0,5 0.4543	0,1,0 0,5 0.5316	0,3,0 0,2 0.6343	0,3,0 0,2 0.7429	0,3,0 0,2 0.8313	0,2,0 1,1 0.8969	0,2,0 1,1 0.9422	0,2,0 1,1 0.9699	0,2,0 1,1 0.9855

Table 3.5 (continued)

No. of blocks	d/σ									
(b)	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
27	0,0,1 0,5 0.4040	0,0,1 0,5 0.4671		0,2,1 0,2 0.6409	0,2,1 0,2 0.7447					
28			0,2,0 0,4 0.5413	0,4,0 0,1 0.6452	0,4,0 0,1 0.7629	0,4,0 0,1 0.8542	0,4,0 0,1 0.9176	0,4,0 0,1 0.9572	0,4,0 0,1 0.9796	0,4,0 0,1 0.9910
29			0,1,1 0,4 0.5458	0,3,1 0,1 0.6556	0,3,1 0,1 0.7691	0,3,1 0,1 0.8572	0,3,1 0,1 0.9188	0,3,1 0,1 0.9575		
30		0,1,0 0,6 0.4692	0,0,2 0,4 0.5477	0,3,0 0,3 0.6675	0,5,0 0,0 0.7761	0,5,0 0,0 0.8689	0,5,0 0,0 0.9302	0,5,0 0,0 0.9661	0,5,0 0,0 0.9850	0,5,0 0,0 0.9939
31	0,0,1 0,6 0.4150	0,0,1 0,6 0.4784	0,2,1 0,3 0.5541	0,2,1 0,3 0.6694	0,4,1 0,0 0.7856	0,4,1 0,0 0.8750	0,4,1 0,0 0.9337	0,4,1 0,0 0.9680	0,4,1 0,0 0.9859	0,4,1 0,0 0.9943
32			0,2,0 0,5 0.5628	0,4,0 0,2 0.6835	0,4,0 0,2 0.7963	0,4,0 0,2 0.8802	0,4,0 0,2 0.9357	0,3,0 1,1 0.9688	0,3,0 1,1 0.9862	0,3,0 1,1 0.9945
33			0,1,1 0,5 0.5640	0,3,1 0,2 0.6889	0,3,1 0,2 0.7980					

Table 3.5 (continued)

No. of blocks (b)	d/σ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
34		0,1,0 0,7 0.4813	0,3,0 0,4 0.5729	0,5,0 0,1 0.6948	0,5,0 0,1 0.8128	0,5,0 0,1 0.8965	0,5,0 0,1 0.9484	0,5,0 0,1 0.9768	0,5,0 0,1 0.9906	0,5,0 0,1 0.9965
35	0,0,1 0,7 0.4240	0,0,1 0,7 0.4877	0,2,1 0,4 0.5771	0,4,1 0,1 0.7028	0,4,1 0,1 0.8174	0,4,1 0,1 0.8986	0,4,1 0,1 0.9492	0,4,1 0,1 0.9771		
36			0,2,0 0,6 0.5801	0,4,0 0,3 0.7124	0,6,0 0,0 0.8245	0,6,0 0,0 0.9076	0,6,0 0,0 0.9565	0,6,0 0,0 0.9817	0,6,0 0,0 0.9931	0,6,0 0,0 0.9976
37			0,3,1 0,3 0.5860	0,3,1 0,3 0.7143	0,5,1 0,0 0.8312	0,5,1 0,0 0.9115	0,5,1 0,0 0.9585	0,5,1 0,0 0.9826	0,5,1 0,0 0.9935	0,5,1 0,0 0.9978
38		0,1,0 0,8 0.4913	0,3,0 0,5 0.5941	0,5,0 0,2 0.7269	0,5,0 0,2 0.8390	0,5,0 0,2 0.9151	0,5,0 0,2 0.9599	0,4,0 1,1 0.9831	0,4,0 1,1 0.9937	0,4,0 1,1 0.9979
39	0,0,1 0,8 0.4316	0,0,1 0,8 0.4955	0,2,1 0,5 0.5958	0,4,1 0,2 0.7313	0,4,1 0,2 0.8405					
40			0,4,0 0,4 0.6041	0,6,0 0,1 0.7377	0,6,0 0,1 0.8525	0,6,0 0,1 0.9266	0,6,0 0,1 0.9677	0,6,0 0,1 0.9874	0,6,0 0,1 0.9956	0,6,0 0,1 0.9987

Table 3.5 (continued)

No. of blocks (b)	d/σ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
41			0,3,1 0,4 0.6079	0,5,1 0,1 0.7440	0,5,1 0,1 0.8558	0,5,1 0,1 0.9281	0,5,1 0,1 0.9683	0,5,1 0,1 0.9876	0,5,1 0,1 0.9957	
42		0,1,0 0,9 0.4999	0,3,0 0,6 0.6115	0,5,0 0,3 0.7518	0,7,0 0,0 0.8624	0,7,0 0,0 0.9348	0,7,0 0,0 0.9729	0,7,0 0,0 0.9901	0,7,0 0,0 0.9968	0,7,0 0,0 0.9991
43	0,0,1 0,9 0.4380	0,0,1 0,9 0.5022	0,4,1 0,3 0.6169	0,4,1 0,3 0.7536	0,6,1 0,0 0.8672	0,6,1 0,0 0.9373	0,6,1 0,0 0.9741	0,6,1 0,0 0.9906	0,6,1 0,0 0.9970	
44			0,4,0 0,5 0.6242	0,6,0 0,2 0.7648	0,6,0 0,2 0.8730	0,6,0 0,2 0.9398	0,6,0 0,2 0.9750	0,6,0 0,2 0.9909	0,6,0 0,2 0.9971	0,5,0 1,1 0.9992
45		0,1,1 0,8 0.5026	0,3,1 0,5 0.6261	0,5,1 0,2 0.7683	0,5,1 0,2 0.8742	0,5,1 0,2 0.9399				
46		0,1,0 0,10 0.5072	0,5,0 0,4 0.6340	0,7,0 0,1 0.7748	0,7,0 0,1 0.8839	0,7,0 0,1 0.9480	0,7,0 0,1 0.9798	0,7,0 0,1 0.9932	0,7,0 0,1 0.9980	0,7,0 0,1 0.9995
47	0,0,1 0,10 0.4437	0,0,1 0,10 0.5079	0,4,1 0,4 0.6374	0,6,1 0,1 0.7798	0,6,1 0,1 0.8863	0,6,1 0,1 0.9491	0,6,1 0,1 0.9802	0,6,1 0,1 0.9933		

Table 3.6

Optimal Designs[1/] and Associated Confidence Coefficient (P) as a Function of

b and d/σ for $p = 5$, $k = 3$

No. of blocks (b)	d/σ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
5	1,0,0 0,0,0 0.0428	1,0,0 0,0,0 0.0575	1,0,0 0,0,0 0.0757	1,0,0 0,0,0 0.0977	1,0,0 0,0,0 0.1238	1,0,0 0,0,0 0.1540	1,0,0 0,0,0 0.1884	1,0,0 0,0,0 0.2267	1,0,0 0,0,0 0.2685	1,0,0 0,0,0 0.3134
7	0,1,0 0,0,0 0.1487	0,1,0 0,0,0 0.1894	0,1,0 0,0,0 0.2361	0,1,0 0,0,0 0.2884	0,1,0 0,0,0 0.3453	0,1,0 0,0,0 0.4056	0,1,0 0,0,0 0.4679	0,1,0 0,0,0 0.5307	0,1,0 0,0,0 0.5925	0,1,0 0,0,0 0.6517
10	0,0,1 0,0,0 0.2123	0,0,1 0,0,0 0.2643	0,0,1 0,0,0 0.3220	0,0,1 0,0,0 0.3842	0,0,1 0,0,0 0.4494	0,0,1 0,0,0 0.5157	0,0,1 0,0,0 0.5815	0,0,1 0,0,0 0.6448	0,0,1 0,0,0 0.7042	0,0,1 0,0,0 0.7585
14	0,0,0 0,1,0 0.2545	0,0,0 0,1,0 0.3126	0,0,0 0,1,0 0.3756	0,0,0 0,1,0 0.4420	0,0,0 0,1,0 0.5100	0,0,0 0,1,0 0.5776	0,2,0 0,0,0 0.6459	0,2,0 0,0,0 0.7236	0,2,0 0,0,0 0.7913	0,2,0 0,0,0 0.8477
17	0,1,0 0,0,1 0.2850	0,1,0 0,0,1 0.3467	0,1,0 0,0,1 0.4127	0,1,0 0,0,1 0.4812	0,1,0 0,0,1 0.5502	0,1,1 0,0,0 0.6286	0,1,1 0,0,0 0.7110	0,1,0 1,0,0 0.7843	0,1,0 1,0,0 0.8505	0,1,0 1,0,0 0.9009
20	0,0,1 0,0,1 0.3083	0,0,1 0,0,1 0.3724	0,0,1 0,0,1 0.4402	0,0,1 0,0,1 0.5098	0,0,0 1,0,1 0.5861	0,0,0 1,0,1 0.6742	0,0,1 1,0,0 0.7638	0,0,1 1,0,0 0.8373	0,0,1 1,0,0 0.8935	0,0,0 2,0,0 0.9345

[1/] The matrix in each cell is $\begin{Bmatrix} \hat{f}_1, \hat{f}_2, \hat{f}_3 \\ \hat{f}_4, \hat{f}_5, \hat{f}_6 \end{Bmatrix}$ where $\hat{D} = \bigcup_{i=1}^{6} \hat{f}_i D_i$ with $b = \sum_{i=1}^{6} \hat{f}_i b_i$ is the optimal design for the given value of b and d/σ.

Table 3.6 (continued)

No. of blocks	d/σ									
(b)	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
24	0,0,0 0,1,1 0.3268	0,0,0 0,1,1 0.3926	0,0,0 0,1,1 0.4616	0,0,0 0,1,1 0.5317	0,2,0 0,0,1 0.6206	0,0,0 1,1,0 0.7197	0,2,0 1,0,0 0.8061	0,2,0 1,0,0 0.8764	0,2,0 1,0,0 0.9259	0,2,0 1,0,0 0.9583
27	0,1,0 0,0,2 0.3420	0,1,0 0,0,2 0.4090	0,1,0 0,0,2 0.4788	0,1,1 0,0,1 0.5545	0,1,0 1,0,1 0.6562	0,1,1 1,0,0 0.7578	0,1,1 1,0,0 0.8426	0,1,0 2,0,0 0.9054	0,1,0 2,0,0 0.9478	0,1,0 2,0,0 0.9731
30	0,0,1 0,0,2 0.3547	0,0,1 0,0,2 0.4227	0,0,1 0,0,2 0.4930	0,0,1 1,0,1 0.5768	0,0,0 2,0,1 0.6880	0,0,1 2,0,0 0.7904	0,0,1 2,0,0 0.8727	0,0,1 2,0,0 0.9286	0,0,1 2,0,0 0.9629	0,0,0 3,0,0 0.9824
34	0,0,0 0,1,2 0.3655	0,0,0 0,1,2 0.4342	0,0,0 0,1,2 0.5049	0,0,0 1,1,1 0.6009	0,2,1 1,0,0 0.7167	0,2,1 1,0,0 0.8202	0,2,0 2,0,0 0.8965	0,2,0 2,0,0 0.9461	0,2,0 2,0,0 0.9742	0,2,0 2,0,0 0.9887
37	0,1,0 0,0,3 0.3749	0,1,0 0,0,3 0.4442	0,1,0 0,0,3 0.5152	0,1,1 1,0,1 0.6237	0,1,0 2,0,1 0.7446	0,1,1 2,0,0 0.8458	0,1,1 2,0,0 0.9161	0,1,0 3,0,0 0.9590	0,1,0 3,0,0 0.9819	0,1,0 3,0,0 0.9927
40	0,0,1 0,0,3 0.3831	0,0,1 0,0,3 0.4529	0,0,0 1,0,3 0.5267	0,0,1 2,0,1 0.6451	0,0,0 3,0,1 0.7696	0,0,1 3,0,0 0.8674	0,0,1 3,0,0 0.9324	0,0,1 3,0,0 0.9690	0,0,1 3,0,0 0.9872	0,0,0 4,0,0 0.9952
44	0,0,0 0,1,3 0.3904	0,0,0 0,1,3 0.4606	0,0,0 1,1,2 0.5384	0,2,1 1,0,1 0.6667	0,2,1 2,0,0 0.7918	0,2,1 2,0,0 0.8864	0,2,0 3,0,0 0.9453	0,2,0 3,0,0 0.9766	0,2,0 3,0,0 0.9910	0,2,0 3,0,0 0.9969

Table 3.6 (continued)

No. of blocks (b)	d/σ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
47	0,1,0 0,0,4 0.3970	0,1,0 0,0,4 0.4674	0,1,0 1,0,3 0.5517	0,1,1 2,0,1 0.6869	0,1,0 3,0,1 0.8126	0,1,1 3,0,0 0.9028	0,1,1 3,0,0 0.9557	0,1,0 4,0,0 0.9822	0,1,0 4,0,0 0.9937	0,1,0 4,0,0 0.9980
50	0,0,1 0,0,4 0.4029	0,0,1 0,0,4 0.4735	0,0,2 1,0,2 0.5648	0,0,1 3.0,1 0.7059	0,0,0 4,0,1 0.8315	0,0,1 4,0,0 0.9167	0,0,1 4,0,0 0.9643	0,0,1 4,0,0 0.9866	0,0,1 4,0,0 0.9955	0,0,1 4,0,0 0.9987
54	0,0,0 0,1,4 0.4082	0,0,0 0,1,4 0.4791	0,2,0 1,0,3 0.5776	0,2,1 2,0,1 0.7242	0,2,1 3,0,0 0.8483	0,2,1 3,0,0 0.9287	0,2,0 4,0,0 0.9711	0,2,0 4,0,0 0.9898	0,2,0 4,0,0 0.9969	0,2,0 4,0,0 0.9991
57	0,1,0 0,0,5 0.4131	0,1,0 0,0,5 0.4841	0,1,0 2,0,3 0.5910	0,1,1 3,0,1 0.7416	0,1,0 4,0,1 0.8635	0,1,1 4,0,0 0.9391	0,1,1 4,0,0 0.9766	0,1,0 5,0,0 0.9923	0,1,0 5,0,0 0.9978	0,1,0 5,0,0 0.9994
60	0,0,1 0,0,5 0.4175	0,0,1 0,0,5 0.4888	0,0,2 2,0,2 0.6039	0,0,1 4,0,1 0.7578	0,0,0 5,0,1 0.8775	0,0,1 5,0,0 0.9479	0,0,1 5,0,0 0.9812	0,0,1 5,0,0 0.9942	0,0,1 5,0,0 0.9984	0,0,1 5,0,0 0.9996
64	0,0,0 0,1,5 0.4216	0,0,0 0,1,5 0.4930	0,2,0 2,0,3 0.6165	0,2,1 3,0,1 0.7732	0,2,1 4,0,0 0.8899	0,2,1 4,0,0 0.9554	0,2,0 5,0,0 0.9848	0,2,0 5,0,0 0.9956	0,2,0 5,0,0 0.9989	0,2,0 5,0,0 0.9998
67	0,1,0 0,0,6 0.4254	0,1,0 0,0,6 0.4969	0,1,0 3,0,3 0.6293	0,1,1 4,0,1 0.7879	0,1,0 5,0,1 0.9010	0,1,1 5,0,0 0.9619	0,1,1 5,0,0 0.9877	0,1,0 6,0,0 0.9966	0,1,0 6,0,0 0.9992	0,1,0 6,0,0 0.9998

Table 3.7

Admissible Designs for $p = 4$, $k = 3$

which are not optimal for any d/σ.

No. of blocks (b)	D_1	D_2	D_3	D_4	D_5
28	0	1	0	1	3
30	0	2	0	1	2
32	0	1	0	1	4

The following table exhibits for $p = 4$, $k = 3$, $b = 28$ the behavior of the confidence coefficients for three admissible designs (the one that is not optimal for any d/σ, and the two that are "adjacent" to it in terms of τ^2 and ρ) as a function of d/σ.

Table 3.8

Confidence Coefficients Associated with Three Admissible Designs

for $p = 4$, $k = 3$, $b = 28$

Design	τ^2	ρ	d/σ					
			0.394	0.395	0.396	0.397	0.398	0.399
$\begin{pmatrix} 0,2,0 \\ 0,4 \end{pmatrix}$	0.1739	0.625	0.63746	0.63876	0.64005	0.64134M	0.64263M	0.64392M
$\begin{pmatrix} 0,1,0 \\ 1,3 \end{pmatrix}$	0.1595	0.563	0.63788	0.63894	0.64000	0.64106	0.64212	0.64317
$\begin{pmatrix} 0,4,0 \\ 0,1 \end{pmatrix}$	0.1250	0.333	0.63832M	0.63932M	0.64032M	0.64131	0.64230	0.64329

*The maximum confidence coefficient in each column is marked with an M; the associated design is the optimal one for that value of d/σ.

Optimal designs to achieve a specified confidence coefficient $1-\alpha$ are given as a function of d/σ for $1-\alpha = 0.75, 0.80, 0.85, 0.90, 0.95,$ and 0.99, and $d/\sigma = 0.2(0.2)2.0$ for $p = 4, k = 3$ in Table 3.9, and for $p = 5, k = 3$ in Table 3.10. These designs were found by a complete computer search among all admissible designs; the entry in Table 3.9 for $1-\alpha = 0.99$, $d/\sigma = 0.2$ was too costly to obtain.

Remark 3.4: We note from Tables 3.9 and 3.10 that for fixed small d/σ and $1-\alpha \to 1$, the optimal design for $p = 4, k = 3$ essentially employs only replications of D_2 and D_5 while the optimal design for $p = 5, k = 3$ essentially employs only replications of D_4 and D_6. Both D_2 and D_4 are BIB designs among the p test treatments augmented by a control treatment in each block, while both D_5 and D_6 are BIB designs among the p test treatments with no controls. Recalling that BIB designs have the property of maximizing the λ-value (for given parameters of the design) we note that the optimal designs used here tend to be those with large values of λ_1.

4. CONCLUDING REMARKS

As mentioned in Section 1, the key unsolved problem relating to BTIB designs is that of the enumeration of all generator designs for a particular (p,k). For as can be seen from Definition 2.6, it is only when the full set of generator designs is known for a particular (p,k) that one can construct the minimal complete class of generator designs. And the designs in that class must all be known in order to verify that a particular design is truly optimal for a given (p,k) and specified

Table 3.9

Optimal Design[1] to Achieve a Specified Confidence Coefficient

as a Function of d/σ

for $p = 4$, $k = 3$

Confidence Coefficient $(1-\alpha)$	d/σ									
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
0.99		b=168 0,28,0 0,0	b=76 0,12,0 0,1	b=42 0,7,0 0,0	b=28 0,4,0 0,1	b=22 0,3,0 0,1	b=16 0,1,0 1,0	b=12 0,2,0 0,0	b=10 0,0,0 1,0	b=10 0,0,0 1,0
0.95	b=420 0,68,0 0,3	b=106 0,17,0 0,1	b=48 0,8,0 0,0	b=28 0,4,0 0,1	b=18 0,3,0 0,0	b=12 0,2,0 0,0	b=10 0,0,0 1,0	b=10 0,0,0 1,0	b=6 0,1,0 0,0	b=6 0,1,0 0,0
0.90	b=312 0,48,0 0,6	b=78 0,13,0 0,0	b=36 0,6,0 0,0	b=22 0,3,0 0,1	b=13 0,1,1 0,0	b=10 0,0,0 1,0	b=10 0,0,0 1,0	b=6 0,1,0 0,0	b=6 0,1,0 0,0	b=6 0,1,0 0,0
0.85	b=248 0,36,0 0,8	b=62 0,9,0 0,2	b=28 0,4,0 0,1	b=16 0,2,0 0,1	b=12 0,2,0 0,0	b=10 0,0,0 1,0	b=6 0,1,0 0,0	b=6 0,1,0 0,0	b=6 0,1,0 0,0	b=6 0,1,0 0,0
0.80	b=202 0,29,0 0,7	b=51 0,6,1 0,2	b=24 0,4,0 0,0	b=13 0,1,1 0,0	b=10 0,1,0 0,1	b=6 0,1,0 0,0	b=6 0,1,0 0,0	b=6 0,1,0 0,0	b=6 0,1,0 0,0	b=4 1,0,0 0,0
0.75	b=167 0,22,1 0,7	b=42 0,5,0 0,3	b=19 0,2,1 0,0	b=12 0,2,0 0,0	b=10 0,1,0 0,1	b=6 0,1,0 0,0	b=6 0,1,0 0,0	b=6 0,1,0 0,0	b=6 0,1,0 0,0	b=4 1,0,0 0,0

[1] The "matrix" in each cell is $\left\{ \begin{matrix} \hat{f}_1, \hat{f}_2, \hat{f}_3 \\ \hat{f}_4, \hat{f}_5 \end{matrix} \right\}$ where $\hat{D} = \bigcup_{i=1}^{5} \hat{f}_i D_i$ with $b = \sum_{i=1}^{5} \hat{f}_i b_i$ is the optimal design for the given value of $1-\alpha$ and d/σ.

Table 3.10

Optimal Design[1/] to Achieve a Specified Confidence Coefficient

as a Function of d/σ

for $p = 5$, $k = 3$

Confidence Coefficient $(1-\alpha)$	d/σ									
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
0.99	b=857 0,1,1 79,0,5	b=217 0,1,1 19,0,1	b=97 0,1,1 8,0,0	b=57 0,1,0 5,0,0	b=37 0,1,0 3,0,0	b=27 0,1,0 2,0,0	b=20 0,0,0 2,0,0	b=17 0,1,0 1,0,0	b=14 0,2,0 0,0,0	b=10 0,0,0 1,0,0
0.95	b=550 0,0,1 49,0,5	b=140 0,0,1 12,0,1	b=64 0,2,1 4,0,0	b=37 0,1,0 3,0,0	b=24 0,2,0 1,0,0	b=17 0,1,0 1,0,0	b=14 0,2,0 0,0,0	b=10 0,0,0 1,0,0	b=10 0,0,0 1,0,0	b=7 0,1,0 0,0,0
0.90	b=417 0,1,1 35,0,5	b=107 0,1,1 8,0,1	b=47 0,1,1 3,0,0	b=27 0,1,0 2,0,0	b=17 0,1,0 1,0,0	b=14 0,2,0 0,0,0	b=10 0,0,0 1,0,0	b=7 0,1,0 0,0,0	b=7 0,1,0 0,0,0	b=7 0,1,0 0,0,0
0.85	b=337 0,1,0 27,0,6	b=87 0,1,1 6,0,1	b=40 0,0,1 3,0,0	b=24 0,2,0 1,0,0	b=17 0,1,0 1,0,0	b=10 0,0,0 1,0,0	b=10 0,0,0 1,0,0	b=7 0,1,0 0,0,0	b=7 0,1,0 0,0,0	b=7 0,1,0 0,0,0
0.80	b=280 0,0,1 21,0,6	b=70 0,0,1 5,0,1	b=34 0,2,1 1,0,0	b=20 0,0,1 1,0,0	b=14 0,2,0 0,0,0	b=10 0,0,0 1,0,0	b=7 0,1,0 0,0,0	b=7 0,1,0 0,0,0	b=7 0,1,0 0,0,0	b=7 0,1,0 0,0,0
0.75	b=237 0,1,1 16,0,6	b=60 0,0,1 4,0,1	b=27 0,1,1 1,0,1	b=17 0,1,0 1,0,0	b=10 0,0,1 0,0,0	b=7 0,1,0 0,0,0	b=7 0,1,0 0,0,0	b=7 0,1,0 0,0,0	b=7 0,1,0 0,0,0	b=5 1,0,0 0,0,0

[1/] The matrix in each cell is $\left\{\begin{matrix} \hat{f}_1, \hat{f}_2, \hat{f}_3 \\ \hat{f}_4, \hat{f}_5, \hat{f}_6 \end{matrix}\right\}$ where $\hat{D} = \bigcup_{i=1}^{6} \hat{f}_i D_i$ with $b = \sum_{i=1}^{6} \hat{f}_i b_i$ is the optimal design for the given value of $1-\alpha$ and d/σ.

d/σ. As stated in Remark 2.7 we conjecture that we have indeed found the minimal complete class of generator designs for $p = 4, k = 3$ and $p = 5, k = 3$, and hence that the designs which we give in Tables 3.5 and 3.6 are the optimal ones. However, our conjecture remains to be proved, and we suspect that the proof will be difficult. In any event, we believe that if one or more designs are missing from our conjectured minimal complete classes for $(p = 4, k = 3)$ or $(p = 5, k = 3)$, then the incremental gain that would be achieved by using the full set in place of our set would be very small.

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APPENDIX

Proofs of C-inadmissibility of certain designs

A.1 Proof for $p = 4$, $k = 3$ of C-inadmissibility of D_1 for all $b \neq 4$

We note that D_1 is admissible for $b = 4$. We now consider unions of D_1 with the designs $D_1,\ldots,D_5$ of Table 3.1. From Table A.1 we see that (i) $D_1 \cup D_1$ is inadmissible w.r.t. D_3 but not S-inadmissible, (ii) $D_1 \cup D_2$ is inadmissible w.r.t. D_4 but not S-inadmissible, (iii) $D_1 \cup D_3$ is S-inadmissible w.r.t. D_4, (iv) $D_1 \cup D_4$ is S-inadmissible w.r.t. $D_2 \cup D_2$, and (v) $D_1 \cup D_5$ is S-inadmissible w.r.t. D_3. We thus need only consider further cases (i) and (ii) since these do not lead to S-inadmissible designs

Table A.1

D_A	$\lambda_0^{(A)}$	$\lambda_1^{(A)}$	b_A	τ_A^2	ρ_A	D_B	$\lambda_0^{(B)}$	$\lambda_1^{(B)}$	b_B	τ_B^2	ρ_B
$D_1 \cup D_1$	4	0	8	0.7500	0.000	D_3	2	2	7	0.6000	0.500
$D_1 \cup D_1 \cup D_3$	6	2	15	--	--	$D_2 \cup D_2$	6	2	12	--	--
$D_1 \cup D_1 \cup D_5$	4	2	12	--	--	D_4	4	2	10	--	--
$D_1 \cup D_2$	5	1	10	0.4000	0.250	D_4	4	2	10	0.3750	0.333
$D_1 \cup D_2 \cup D_3$	7	3	17	--	--	$D_2 \cup D_4$	7	3	16	--	--
$D_1 \cup D_2 \cup D_5$	5	3	14	--	--	$D_2 \cup D_3$	5	3	13	--	--
$D_1 \cup D_3$	4	2	11	--	--	D_4	4	2	10	--	--
$D_1 \cup D_4$	6	2	14	--	--	$D_2 \cup D_2$	6	2	12	--	--
$D_1 \cup D_5$	2	2	8	--	--	D_3	2	2	7	--	--

We first consider case (i) and characterize the class of BTIB designs D for which $(D_1 \cup D_1) \cup D$ is inadmissible w.r.t. $D_3 \cup D$. Let (λ_0, λ_1) be the parameters associated with D; for $D_1 \cup D_1$ we have $\lambda_0 = 4$, $\lambda_1 = 0$, $b = 8$ while for D_3 we have $\lambda_0 = 2$, $\lambda_1 = 2$, $b = 7$. For $(D_1 \cup D_1) \cup D$ to be inadmissible with respect to $D_3 \cup D$ we must have

$$\rho\{(D_1 \cup D_1) \cup D\} \leqq \rho\{D_3 \cup D\} \tag{A.1}$$

and

$$\tau^2\{(D_1 \cup D_1) \cup D\} \geqq \tau^2\{D_3 \cup D\}. \tag{A.2}$$

For case (i) we see that (A.1) and (A.2) become

$$\frac{\lambda_1}{\lambda_0+\lambda_1+4} \leqq \frac{\lambda_1+2}{\lambda_0+\lambda_1+4} \tag{A.3}$$

and

$$\frac{\lambda_0+\lambda_1+4}{(\lambda_0+4)(\lambda_0+4\lambda_1+4)} \geqq \frac{\lambda_0+\lambda_1+4}{(\lambda_0+2)(\lambda_0+4\lambda_1+10)} \tag{A.4a}$$

or

$$\lambda_0 - 2\lambda_1 + 1 \geqq 0, \tag{A.4b}$$

respectively. Now (A.3) always holds; (A.4b) holds for D_1, D_2, D_4 (or any unions of their replications) but not for D_3 and D_5. However, $(D_1 \cup D_1) \cup D_3$ is S-inadmissible w.r.t. $D_2 \cup D_2$ while $(D_1 \cup D_1) \cup D_5$ is S-inadmissible w.r.t. D_4.

We next consider case (ii), and proceed as above. For $D_1 \cup D_2$ we have $\lambda_0 = 5$, $\lambda_1 = 1$, $b = 10$ while for D_4 we have $\lambda_0 = 4$, $\lambda_1 = 2$, $b = 10$. For $(D_1 \cup D_2) \cup D$ to be inadmissible w.r.t. $D_4 \cup D$ we must have

$$\rho\{(D_1 \cup D_2) \cup D\} \leqq \rho\{D_4 \cup D\} \tag{A.5}$$

and

$$\tau^2\{(D_1 \cup D_2) \cup D\} \geqq \tau^2\{D_4 \cup D\} \tag{A.6}$$

with at least one inequality strict. For case (ii) we see that (A.5) and (A.6) become

$$\frac{\lambda_1+1}{\lambda_0+\lambda_1+6} \geqq \frac{\lambda_1+2}{\lambda_0+\lambda_1+6} \tag{A.7}$$

and

$$\frac{\lambda_0+\lambda_1+6}{(\lambda_0+5)(\lambda_0+4\lambda_1+9)} \geqq \frac{\lambda_0+\lambda_1+6}{(\lambda_0+4)(\lambda_0+4\lambda_1+12)} \tag{A.8a}$$

or

$$2\lambda_0 - 4\lambda_1 + 3 \geqq 0, \tag{A.8b}$$

respectively. Now (A.7) always holds; (A.8b) holds for D_1, D_2, D_4 (or any unions of their replications) but not for D_3 or D_5. However, $(D_1 \cup D_2) \cup D_3$ is S-inadmissible w.r.t. $D_2 \cup D_4$ and $(D_1 \cup D_2) \cup D_5$ is S-inadmissible w.r.t. $D_2 \cup D_3$. This completes the proof of the C-inadmissibility of D_1 w.r.t. $\{D_2,D_3,D_4,D_5\}$ for $p = 4$, $k = 3$.

A.2 Proof for p = 5, k = 3 of C-inadmissibility of D_7 for all b

We note that D_7 is unimplementable by itself. We now consider unions of D_7 with the designs $D_1,\ldots,D_6$. From Table A.2 we see that (i) $D_7 \cup D_1$ is S-inadmissible w.r.t. D_3, (ii) $D_7 \cup D_2$ is S-inadmissible w.r.t. D_5, (iii) $D_7 \cup D_3$ is S-inadmissible w.r.t. $D_2 \cup D_6$, (iv) $D_7 \cup D_4$ is S-inadmissible w.r.t. $D_2 \cup D_3$, (v) $D_7 \cup D_5$ is S-inadmissible w.r.t. $D_3 \cup D_6$, and (vi) $D_7 \cup D_6$ is nonimplementable. Thus D_7 is C-inadmissible w.r.t. $\{D_1,D_2,D_3,D_4,D_5,D_6\}$ for all b.

Table A.2

D_A	$\lambda_0^{(A)}$	$\lambda_1^{(A)}$	b_A	D_B	$\lambda_0^{(B)}$	$\lambda_1^{(B)}$	b_B
$D_7 \cup D_1$	2	2	13	D_3	2	2	10
$D_7 \cup D_2$	2	3	15	D_5	2	3	14
$D_7 \cup D_3$	2	4	18	$D_2 \cup D_6$	2	4	17
$D_7 \cup D_4$	4	3	18	$D_2 \cup D_3$	4	3	17
$D_7 \cup D_5$	2	5	22	$D_3 \cup D_6$	2	5	20

Note that $D_7 \cup D_6$ is nonimplementable since it contains no control treatments.

A.3 Proof for $p = 5$, $k = 3$ of C-inadmissibility of D_1 for all $b \neq 5,15$

We note that D_1 is admissible for $b = 5$, and $D_1 \cup D_4$ is admissible for $b = 15$. We now consider unions of D_1 with the designs D_1, D_2, D_3, D_5, D_6, and unions of $D_1 \cup D_4$ with designs $D_1,\ldots,D_6$. From Table A.3 we see that (i) $D_1 \cup D_1$ is S-inadmissible w.r.t. D_4, (ii) $D_1 \cup D_2$ is S-inadmissible w.r.t. D_4, (iii) $D_1 \cup D_3$ is S-inadmissible w.r.t. $D_2 \cup D_2$, (iv) $D_1 \cup D_5$ is S-inadmissible w.r.t. $D_2 \cup D_3$, (v) $D_1 \cup D_6$ is S-inadmissible w.r.t. D_5, (vi) $(D_1 \cup D_4) \cup D_1$ is S-inadmissible w.r.t. $D_4 \cup D_4$, (vii) $(D_1 \cup D_4) \cup D_2$ is S-inadmissible w.r.t. $D_4 \cup D_4$, (viii) $(D_1 \cup D_4) \cup D_3$ is S-inadmissible w.r.t. $D_2 \cup D_2 \cup D_4$, (ix) $(D_1 \cup D_4) \cup D_4$ is inadmissible w.r.t. $D_2 \cup D_2 \cup D_4$ but not S-inadmissible, (x) $(D_1 \cup D_4) \cup D_5$ is S-inadmissible w.r.t. $D_2 \cup D_3 \cup D_4$, and (xi) $(D_1 \cup D_4) \cup D_6$ is S-inadmissible w.r.t. $D_4 \cup D_5$. We thus need consider only case (ix) since it is the only one that does not lead to an S-inadmissible design. Proceeding as in Section A.1, for

Table A.3

D_A	$\lambda_0^{(A)}$	$\lambda_1^{(A)}$	b_A	τ_A^2	ρ_A	D_B	$\lambda_0^{(B)}$	$\lambda_1^{(B)}$	b_B	τ_B^2	ρ_B
$D_1 \cup D_1$	4	0	10	--	--	D_4	4	1	10	--	--
$D_1 \cup D_2$	4	1	12	--	--	D_4	4	1	10	--	--
$D_1 \cup D_3$	4	2	15	--	--	$D_2 \cup D_2$	4	2	14	--	--
$D_1 \cup D_5$	4	3	19	--	--	$D_2 \cup D_3$	4	3	17	--	--
$D_1 \cup D_6$	2	3	15	--	--	D_5	2	3	14	--	--
$D_1 \cup D_4 \cup D_1$	8	1	20	--	--	$D_4 \cup D_4$	8	2	20	--	--
$D_1 \cup D_4 \cup D_2$	8	2	22	--	--	$D_4 \cup D_4$	8	2	20	--	--
$D_1 \cup D_4 \cup D_3$	8	3	25	--	--	$D_2 \cup D_2 \cup D_4$	8	3	24	--	--
$D_1 \cup D_4 \cup D_4$	10	2	25	0.1800	0.167	$D_2 \cup D_2 \cup D_4$	8	3	24	0.1793	0.273
$D_1 \cup D_4 \cup D_5$	8	4	29	--	--	$D_2 \cup D_3 \cup D_4$	8	4	27	--	--
$D_1 \cup D_4 \cup D_6$	6	4	25	--	--	$D_4 \cup D_5$	6	4	24	--	--

$(D_1 \cup D_4) \cup D_4$ we have $\lambda_0 = 10$, $\lambda_1 = 2$, $b = 25$ while for $D_2 \cup D_2 \cup D_4$ we have $\lambda_0 = 8$, $\lambda_1 = 3$, $b = 24$. For $(D_1 \cup D_4) \cup D_4 \cup D$ to be inadmissible with respect to $D_2 \cup D_2 \cup D_4 \cup D$ we must have

$$\rho\{D_1 \cup D_4 \cup D_4 \cup D\} \leqq \rho\{D_2 \cup D_2 \cup D_4 \cup D\} \tag{A.9}$$

and

$$\tau^2\{D_1 \cup D_4 \cup D_4 \cup D\} \geqq \tau^2\{D_2 \cup D_2 \cup D_4 \cup D\}. \tag{A.10}$$

For case (ix) we see that (A.9) and (A.10) become

$$\frac{\lambda_1+2}{\lambda_0+\lambda_1+12} \leqq \frac{\lambda_1+3}{\lambda_0+\lambda_1+11} \tag{A.11}$$

and

$$\frac{\lambda_0+\lambda_1+12}{(\lambda_0+10)(\lambda_0+5\lambda_1+20)} \geqq \frac{\lambda_0+\lambda_1+11}{(\lambda_0+8)(\lambda_0+5\lambda_1+23)}\ , \tag{A.12}$$

respectively. Now (A.11) always holds; (A.12) holds for D_1 and D_4 (or any unions of their replications) but not for D_2, D_3, D_5 and D_6. *However*, $(D_1 \cup D_4) \cup D_4 \cup D_2$ is S-inadmissible w.r.t., $D_4 \cup D_4 \cup D_4$, $(D_1 \cup D_4) \cup D_4 \cup D_3$ is S-inadmissible w.r.t. $D_2 \cup D_2 \cup D_4 \cup D_4$, $(D_1 \cup D_4) \cup D_4 \cup D_5$ is S-inadmissible w.r.t. $D_2 \cup D_3 \cup D_4 \cup D_4$, and $(D_1 \cup D_4) \cup D_4 \cup D_6$ is S-inadmissible w.r.t. $D_4 \cup D_4 \cup D_5$. This completes the proof for $p = 5$, $k = 3$ of the C-inadmissibility of D_1 w.r.t. $\{D_2, D_3, D_4, D_5, D_6\}$ for $b \neq 5, 15$.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper continues the study of balanced treatment incomplete block (BTIB) designs initiated in [1] and [2]. The definitions of inadmissibility and strong inadmissibility introduced in [2] are generalized. The search for optimal designs is restricted to the smaller class of admissible designs obtained using these new definitions. Lists of generator designs, the (conjectured) complete class of generator designs, catalogs of admissible designs, and tables of optimal designs are given for $p = 4$, $k = 3$ and

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$p = 5, k = 3$.

[1] Bechhofer, R.E. and Tamhane, A.C. (1979a). Incomplete block designs for comparing treatments with a control (I): General theory. (Submitted for publication.)

[2] Bechhofer, R.E. and Tamhane, A.C. (1979b). Incomplete block designs for comparing treatments with a control (II): Optimal designs for $p = 2(1)6$, $k = 2$ and $p = 3$, $k = 3$. (Submitted for publication.)